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SENSITIVITY ANALYSIS OF HYDRODYNAMIC STABILITY OPERATORS

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ABSTRACT

The eigenvalue sensitivity for hydrodynamic stability operators is investigated. Classical matrix perturbation techniques as well as the concept of ε -pseudoeigenvalues are applied to show that parts of the spectrum are highly sensitive to small perturbations. Applications are drawn from incompressible plane Couette, trailing line vortex flow and compressible Blasius boundary layer flow. Parametric studies indicate a monotonically increasing effect of the Reynolds number on the sensitivity. The phenomenon of eigenvalue sensitivity is due to the non-normality of the operators and their discrete matrix analogs and may be associated with large transient growth of the corresponding initial value problem.

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1. Introduction

Hydrodynamic stability theory has received a great deal of attention over the past decades (see e.g. [3][13]) and still is a field of active and ongoing research. Its widespread applications range from basic fluid dynamics to magnetohydrodynamics, from geophysics to aeroelasticity, to name only a few.

The starting point is a nonlinear evolution equation for a state vector describing the flow field. A basic state is assumed and a linearization of the nonlinear equation about this mean state results in a linear initial-boundary-value problem for the perturbations Φ of the form $\partial\Phi/\partial t + i\mathcal{L}\Phi = 0$. The linear operator \mathcal{L} depends in general on parameters describing the particular flow problem, such as Reynolds number, Mach number etc.

Linear stability is defined in an asymptotic manner for $t \rightarrow \infty$. If for a given value of the flow parameters any (small) perturbation dies out, the system is said to be linearly stable for this particular parameter value. In the case of exponentially growing perturbations the system is linearly unstable and nonlinear effects will become significant once the initially small perturbations reach sufficient amplitudes.

Eigenvalue analysis is the most dominant tool in determining the linear stability of a system. It is based on the assumption that the time dependence of the perturbations can be expressed in the form $\exp(-i\lambda t)$. This reduces the linear initial-boundary-value problem to an eigenvalue problem $\lambda\Phi = \mathcal{L}\Phi$. If the spectrum of \mathcal{L} consists of any eigenvalues whose imaginary part is positive, the system will support exponentially growing perturbations. If the spectrum of \mathcal{L} is confined to the lower complex half-plane, the system is linearly stable and small perturbations will decay for sufficiently large time. This technique of determining the linear stability has been used almost exclusively during the past decades. It is important, however, to stress that both linear stability and eigenvalue analysis are defined in an asymptotic setting ($t \rightarrow \infty$) and do not provide any information about the transient behavior of the operator \mathcal{L} .

In most cases, the operators involved in hydrodynamic stability problems are non-normal, i.e. although their set of eigenfunctions may be complete, the eigenfunctions do not form an orthogonal basis. Although this has been known for a long time [1], little attention has been paid to its consequences for the use of eigenvalues to describe the temporal behavior of infinitesimal disturbances. Non-normality may result in a large sensitivity of the eigenvalues to small perturbations of the operator. This has often been experienced as unsatisfactory convergence of parts of the spectrum and has been noted by several investigators [6][14][17]. As will be shown in this report, the sensitivity of the eigenvalues reveals more information about the short-time behavior of the operator than do the eigenvalues. Large transient growth is often associated with a high sensitivity of \mathcal{L} and can occur even if the eigenvalue spectrum predicts decay. This growth can take substantial values depending on the departure from normality of the operator or its discrete matrix analog. For this reason, *the study of the sensitivity of hydrodynamic stability operators is as essential as the study of its eigenvalues.*

It has recently been shown [19] that parts of the Orr-Sommerfeld spectrum for plane Poiseuille flow are highly sensitive to small perturbations. Furthermore, even for Reynolds

numbers far less than the critical value of 5772, large transient growth of perturbation energy has been found. For a detailed analysis the reader is referred to [19].

In this paper, we address the issue of eigenvalue sensitivity and discuss the use of eigenvalue analysis for the investigation of transient non-mode-like behavior. Applications are drawn from different fields of fluid dynamics. Although limiting ourselves to examples arising in fluid dynamics, we anticipate similar effects for linear stability problems that involve highly non-normal operators.

This paper is organized as follows. Section 2 presents a brief review of some classical matrix perturbation concepts and introduces the idea of ε -pseudoeigenvalues. Section 3 applies the developed tools to the stability of both incompressible and compressible flows. Section 4 summarizes and concludes this paper.

We would like to stress that this work is *not* concerned with roundoff error analysis. The focus of the present analysis is on the sensitivity of the spectrum as an effect of the non-normality of hydrodynamic stability operators.

2. Basics of Matrix Perturbation Theory

Matrix perturbation theory applied to eigenvalue problems is mainly concerned with the question of relating a small perturbation in the entries of a matrix A to the variation in its eigenvalues. Let E be a matrix with random entries and norm $\|E\| = \varepsilon \ll 1$ and let \tilde{A} be the perturbed matrix $A + E$. Given ε and the (unperturbed) spectrum of A , how is the spectrum of \tilde{A} related to the spectrum of A ? One of the most straightforward estimates is given by linear perturbation theory [21].

Let $A \in \mathcal{C}^{N \times N}$ be a diagonalizable matrix, i.e. $V^{-1}AV = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$. The Bauer-Fike theorem (see [5]) relates the maximum eigenvalue deviation (the spectral variation [21]) to the norm of the perturbation matrix in the following way.

$$\max_k \min_j |\tilde{\lambda}_k - \lambda_j| \leq \|V^{-1}EV\| \leq \kappa(V)\|E\|, \quad (1)$$

where $\tilde{\lambda}$ and λ are eigenvalues of \tilde{A} and A , respectively, and $\kappa(V) = \|V^{-1}\|\|V\|$ denotes the condition number of the eigenvector matrix V . Throughout this study, we will use the 2-norm (Euclidean norm) denoted by $\|x\| = \|x\|_2 = \sqrt{(x, x)}$ with $x \in \mathcal{C}^N$ where (\cdot, \cdot) stands for the associated inner product. The 2-norm generalizes to square matrices in an obvious way. Equation (1) states that, if each perturbed eigenvalue is associated with its closest unperturbed eigenvalue, the maximum deviation in the entire spectrum is bounded by the product of condition number of V and the norm of the perturbation matrix. The condition number $\kappa(V) = 1$ if and only if A is normal, i.e. $A^H A = A A^H$ where H denotes the conjugate transpose. Therefore, for normal matrices the eigenvalues of \tilde{A} will at most deviate from the eigenvalues of A by an amount of $\|E\|$. For non-normal matrices, $\kappa(V)$ can be substantially larger than 1 giving rise to $\mathcal{O}(1)$ deviations even for minute perturbations in the matrix entries.

The expression above bounds the largest deviation of an eigenvalue in the entire spectrum as a function of $\|E\|$, but does not account for the sensitivity of *individual* eigenvalues to small perturbations. A simple perturbation expansion for an individual eigenvalue yields [21]

$$\tilde{\lambda}_k = \lambda_k + \frac{y_k^H E x_k}{y_k^H x_k} + \mathcal{O}(\|E\|^2) \quad \|E\| \ll 1 \quad (2)$$

where x_k and y_k denote the right and left eigenvectors of A corresponding to the eigenvalue λ_k . Neglecting higher order terms and taking the norm on both sides of equation (2), one obtains

$$|\tilde{\lambda}_k - \lambda_k| \leq s(\lambda_k) \|E\| \quad (3)$$

with

$$s(\lambda_k) = \frac{\|y_k\| \|x_k\|}{|y_k^H x_k|}. \quad (4)$$

Because of relation (3) the quantity $s(\lambda_k)$ is known as the condition of an eigenvalue [5] and can be interpreted as a measure of sensitivity to small perturbations in A . Again, for normal matrices $s(\lambda_k) = 1$.

Computing the condition $s(\lambda_k)$ of an eigenvalue λ_k , is a straightforward procedure to obtain a first indication about parts of the spectrum that may experience a high sensitivity to small perturbations.

In an effort to analyze the behavior of non-normal operators and matrices, Trefethen introduced the concept of ε -pseudoeigenvalues [22][23]. Although valid for operators and their discrete matrix analogs we will restrict ourselves to the matrix case only.

Let $\Lambda(A)$ be the spectrum of $A \in \mathbb{C}^{N \times N}$. The ε -pseudospectrum $\Lambda_\varepsilon(A)$ can be defined in two equivalent ways [22].

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \|(zI - A)^{-1}\| \geq \varepsilon^{-1}\} \quad (5)$$

or

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : z \in \Lambda(A + E) \text{ for some } E \text{ with } \|E\| \leq \varepsilon\}. \quad (6)$$

The first definition makes use of the norm of the resolvent $R(z) = (zI - A)^{-1}$ which is continuously defined in the complex plane with the exception of the spectrum $\Lambda(A)$. As z approaches an eigenvalue of A , $R(z)$ will become singular and $\|R(z)\|$ will tend to infinity. Therefore, the point spectrum of A will correspond to the locations $z \in \mathbb{C}$ where the norm of the resolvent will become infinite. For finite but large resolvent norm, z will be defined as an ε -pseudoeigenvalue of A with $\varepsilon = \|R(z)\|^{-1}$.

The second definition makes use of the spectrum of a randomly perturbed matrix. z is considered an ε -pseudoeigenvalue of A if it is an (exact) eigenvalue of a matrix \tilde{A} which is perturbed by a random matrix E of norm ε .

Both definitions define the ε -pseudospectrum as a *region* in the complex plane rather than a discrete set. Trefethen [22][23] has shown that the ε -pseudospectrum reveals more about the behavior of non-normal operators and matrices than does the spectrum alone. We illustrate this in the next section by calculating ε -pseudoeigenvalues of highly non-normal matrices arising in hydrodynamic stability theory. For a proof of the equivalence of definitions (5) and (6) and for a more theoretical background on ε -pseudoeigenvalues, the reader is referred to references [19][22].

It is worth noting that the condition of an eigenvalue $s(\lambda_k)$ can be used to bound the ε -pseudospectrum of A . Combining (6) and (3), we find

$$\Lambda_\varepsilon \subseteq \bigcup_{k=1}^N \{\lambda_k + \Delta_{s(\lambda_k)}\varepsilon\} \quad (7)$$

where Δ_r denotes a ball of radius r . This relation states that the ε -pseudospectrum is enclosed in the union of disks around the eigenvalues λ_k with radius $s(\lambda_k)\varepsilon$. It is important to keep in mind that relation (7) only holds for ε sufficiently small such that higher order terms in equation (2) are negligible. Again, for normal matrices, that is $s(\lambda_k) = 1$, we obtain a sharp bound on the ε -pseudospectrum.

An important implication of the sensitivity of the spectra is the possibility of large transient growth of a disturbance governed by a non-normal operator. To demonstrate this, let us return to the linearized initial-boundary-value problem governing the evolution of infinitesimal perturbations,

$$\frac{\partial \phi}{\partial t} + iL\phi = 0, \quad (8)$$

where ϕ denotes the spatially *discretized* state vector and L now stands for the *discretized* matrix analog of the linear stability operator. Since the non-normality is an operator property, any consistent discretization technique for \mathcal{L} will capture its behavior as long as sufficient resolution is ensured. The solution to (8) is given by

$$\phi = \phi_0 \exp(-itL) \quad (9)$$

with ϕ_0 as the initial condition $\phi(t=0)$. Applying eigenvalue analysis, it is straightforward to show [23] that

$$\exp(-i\sigma t) \leq \|\exp(-itL)\| \leq \kappa(V) \exp(-i\sigma t), \quad (10)$$

where σ stands for the largest imaginary part of the spectrum of L , i.e. the imaginary part of the least stable mode of L .

If L is normal, then $\kappa(V) = 1$ and the behavior of the initial-boundary-value problem is completely determined by the eigenvalues of L . In case L is highly non-normal, i.e. $\kappa(V) \gg 1$, the upper bound on $\|\exp(-itL)\|$ allows for transient growth, i.e. $\|\phi\|/\|\phi_0\| \gg 1$, although the spectrum of L is confined to the lower complex half-plane.

For non-normal matrices, it is therefore necessary to compute the norm of the matrix exponential instead of relying on the spectrum of L to make predictions about the temporal transient behavior as described by the initial-boundary-value problem.

A necessary and sufficient condition for the existence of transient growth can be derived in a straightforward manner from the evolution equation for the norm of the state vector. Assuming that the discretized linear initial value problem is given in the form of equation (8), the evolution equation for the norm of the state vector ϕ reads

$$\frac{\partial}{\partial t} \|\phi\| = \left(\frac{\partial \phi}{\partial t}, \phi \right) + \left(\phi, \frac{\partial \phi}{\partial t} \right) = (-iL\phi, \phi) + (\phi, -iL\phi) = 2\text{Im}[(L\phi, \phi)], \quad (11)$$

where the relation $(f, g) = (g, f)^*$ has been utilized.

It is evident from equation (11) that temporal growth of ϕ measured in the Euclidean norm can occur if and only if the imaginary part of $(L\phi, \phi)$ is positive. The quantity $(L\phi, \phi)$ is known as the field of values or, equivalently, the numerical range of L and is defined as

$$\mathcal{F}(L) = \{z : z = (L\phi, \phi) \text{ with } \|\phi\| = 1\}.$$

It is clear that the field of values of L contains the spectrum of L (see [8]). It can further be shown (e.g. [22]) that for normal matrices, $\mathcal{F}(L)$ is the convex hull of the spectrum whereas for non-normal matrices the field of values can be a considerably larger set than the spectrum. Especially cases in which the field of values $\mathcal{F}(L)$ reaches into the unstable domain, although the spectrum of L is confined to the stable half-plane, are of particular interest. For these cases, transient growth followed by asymptotic decay can be predicted.

The boundary of the field of values gives an immediate indication for transient growth and can easily be calculated using standard techniques [8]. It should be kept in mind, however, that the field of values $\mathcal{F}(L)$ depends on the choice of the scalar product and is an indicator for growth measured in its associated norm.

3. Application to Hydrodynamic Stability Problems

In this section we apply the tools developed in the previous chapter to study the sensitivity of an assortment of matrices resulting from discretized hydrodynamic stability operators. We thereby use the following qualitative and quantitative techniques to illustrate the properties of the spectra under consideration.

- Following the second definition of ε -pseudospectra we display a superposition of 100 spectra of randomly perturbed matrices $\tilde{A} = A + E$. The complex entries of the matrix E are drawn randomly from a normal distribution with mean zero and standard deviation one. The norm of the matrix E is chosen to be ε . The plot then shows ε -pseudoeigenvalues in a “statistical” fashion and gives a first impression of the sensitive parts of the spectrum under consideration.

- For a few eigenvalues the condition defined by (4) is tabulated which gives an upper bound for the eigenvalue deviation (equation (3)) for sufficiently small perturbations. Although the bound might not be sharp, it gives insight into the *relative* sensitivity of eigenvalues to small perturbations.
- An illustrative means to understand the spectrum of a non-normal operator is a contour plot of constant resolvent norm $\|(zI - A)^{-1}\| = 1/\varepsilon$ for various ε . It displays closed, nested curves that confine the eigenvalues of the matrix A if perturbed by a random matrix of norm ε . The numerical realization of this kind of plots is rather time consuming and involves a singular value decomposition on grid points discretizing the region of interest in the complex plane (see [19][22]).

We consider three different flow types to demonstrate spectral properties of non-normal operators. The first example is incompressible, plane Couette flow, i.e. the flow between infinite plates moving in opposite direction. Another application comes from trailing line vortex flow. Both inviscid and viscous disturbances are considered. The final example is taken from compressible Blasius boundary layer flow, i.e. flow in a semi-infinite domain.

We would like to emphasize that, as long as sufficient resolution is ensured, the method of discretizing the stability operator does not influence the qualitative results of this study. The sensitivity of the spectrum is a property of the *operator* that will be captured by its discrete matrix analog.

3.1. Plane Couette Flow

Plane, incompressible Couette flow, i.e. flow between infinite parallel plates moving in opposite direction, can be considered as one of the canonical flow situations. The linear mean velocity profile greatly simplifies theoretical investigations, which made this type of flow particularly accessible to analytical methods. Herron [7] showed that plane Couette flow is linearly stable for all Reynolds numbers.

The problem of transient growth in plane Couette flow has been studied previously by Farrell [4] using variational principles and recently by Reddy & Henningson [18] making use of ε -pseudospectra. In this presentation, it is included for completeness and is meant as an introductory example to familiarize the reader with the analysis of non-normal matrices and their associated characteristics.

The equation governing the evolution of infinitesimal, two-dimensional disturbances in plane Couette flow of the form $v(x, y, t) = \hat{v}(y, t) \exp(i\alpha x)$ can be written as

$$\frac{\partial \hat{v}}{\partial t} + i\mathcal{L}\hat{v} = 0 \quad (12a)$$

$$\hat{v}(\pm 1) = D\hat{v}(\pm 1) = 0, \quad (12b)$$

with

$$\mathcal{L} = (D^2 - \alpha^2)^{-1} [\alpha U(D^2 - \alpha^2) - \frac{1}{iR}(D^2 - \alpha^2)^2] \quad (12c)$$

where \hat{v} denotes the Fourier transformed normal velocity, α represents the streamwise wavenumber, $U = y$ stands for the mean velocity profile and D denotes differentiation in the wall normal direction y . All flow quantities have been non-dimensionalized by the velocity of the moving plates U_0 and the channel half-height h and the Reynolds number R is based on these scaling parameters, i.e. $R = U_0 h / \nu$. The equation above is related to the Orr-Sommerfeld equation which has received a great deal of attention over the past decades [2].

The operator \mathcal{L} is spatially discretized using a hybrid Chebyshev pseudospectral technique and the discretized operator is denoted by L .

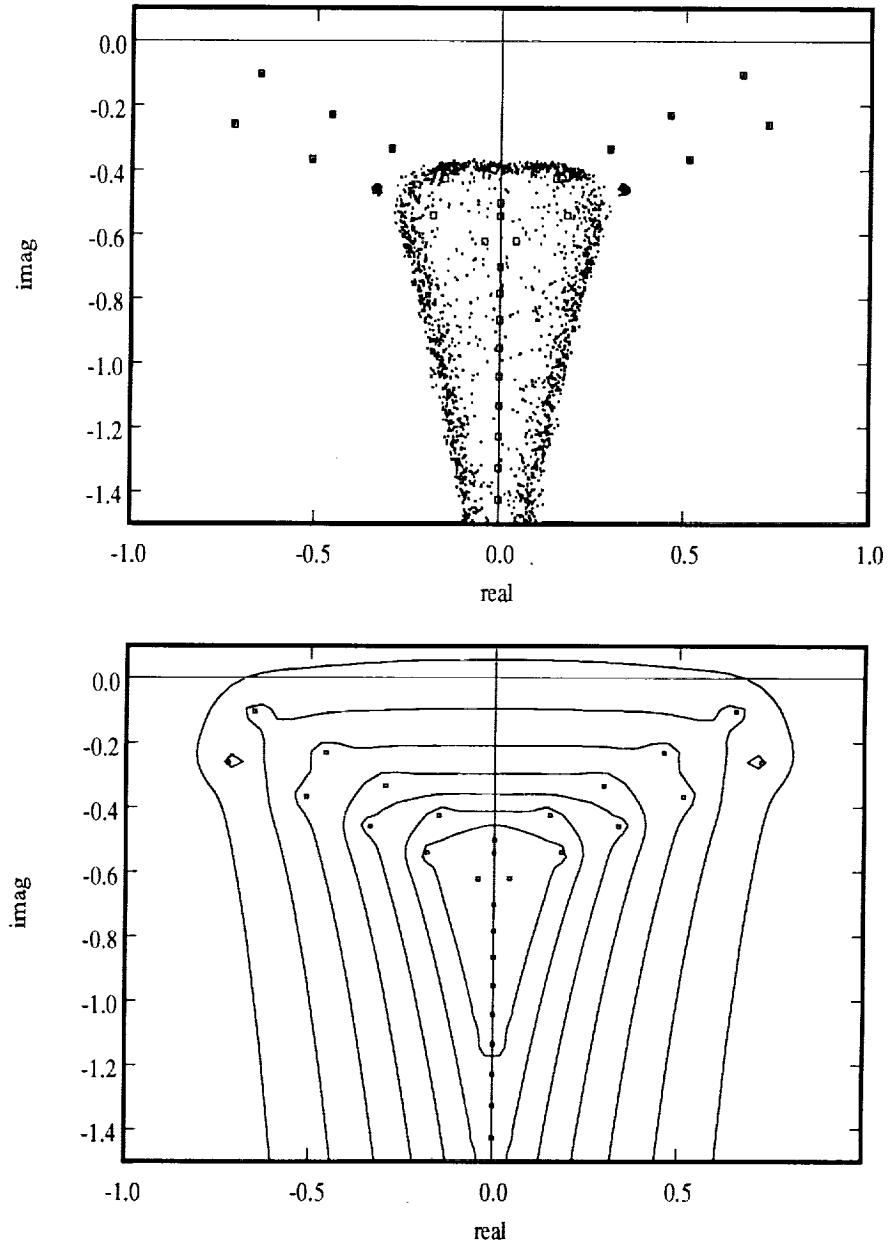


Fig.1. Pseudospectra for plane Couette flow for $\alpha = 1$ and $R = 1500$. (a) Superposition of 100 spectra perturbed by random matrices of norm 10^{-6} . The unperturbed spectrum is displayed by the larger square symbols. (b) Contour plot of constant resolvent norm. The contours represent levels from 10^{-8} (innermost contour) to 10^{-2} (outer contour).

Figure 1a shows the superposition of 100 spectra of L for $\alpha = 1$ and $R = 1500$ perturbed by random matrices of norm 10^{-6} . The unperturbed spectrum is shown by larger squares. It is observed that a perturbation of norm 10^{-6} can cause some eigenvalues to move by a distance of order one. This sensitivity is highest for eigenvalues at the intersection point of the three eigenvalue branches. This observation is also apparent from Table 1 which lists selected eigenvalues together with their condition as defined by (4). It can be seen that the condition is largest for the eigenvalue in the junction point whereas the least stable mode has a rather low condition, which is also reflected in the low sensitivity of the least stable modes in Figure 1a. Figure 1b, which displays the resolvent norm contours together with the unperturbed spectrum, gives the complete picture of the sensitivity properties of the matrix L for the specified parameter values. It is seen that even a perturbation of 10^{-8} has an appreciable effect on the location of the eigenvalues near the intersection point of the three branches. In order to move the least stable modes a distance of order one, a perturbation of norm 10^{-2} is needed which renders them rather insensitive. It should be noted that the 10^{-2} -pseudospectrum extends into the upper half-plane which has implications on the possibility of transient growth. For more information on this issue the reader is referred to [19].

eigenvalue		condition
λ_r	λ_i	
0.723771	-0.259583	$0.178337 \cdot 10^2$
0.653132	-0.102713	$0.289772 \cdot 10^2$
0.336625	-0.457634	$0.328836 \cdot 10^5$
$-0.516156 \cdot 10^{-5}$	-1.42610	$0.833198 \cdot 10^6$
$0.218617 \cdot 10^{-3}$	-0.702198	$0.167807 \cdot 10^9$

Table.1. Selected eigenvalues and conditions for plane Couette flow for $\alpha = 1$ and $R = 1500$.

It is interesting to probe the transient temporal behavior of L . To this end, we calculate the norm of the matrix exponential of L for $\alpha = 1$ and $R = 1500$. Figure 2 shows $\|\exp(-itL)\|$ versus time. Large transient growth up to about $\|\phi\|/\|\phi_0\| \approx 20$ is observed before exponential decay predicted by the eigenvalues sets in. The reason for this growth lies in the non-orthogonality of the set of eigenvectors of L , which in turn is an effect of the non-normality of the respective stability operator. Figure 2 is a striking and instructive demonstration that predictions about the temporal behavior based on the eigenvalues of L can be misleading and that the non-mode-like behavior of the stability operator is not captured by its spectrum.

As explained in section 2, the field of values of the linear stability operator for plane Couette flow can be used as an indicator for transient growth. For the parameter values chosen above, the maximum imaginary part of the field of values has been calculated to be 0.557 which is a measure for the largest growth rate possible.

Although measuring the growth in the 2-norm does reveal the transient behavior of the operator under investigation, a weighted norm has to be used to draw conclusions about the amplification of physical flow quantities such as perturbation energy; for details the reader is referred to [19].

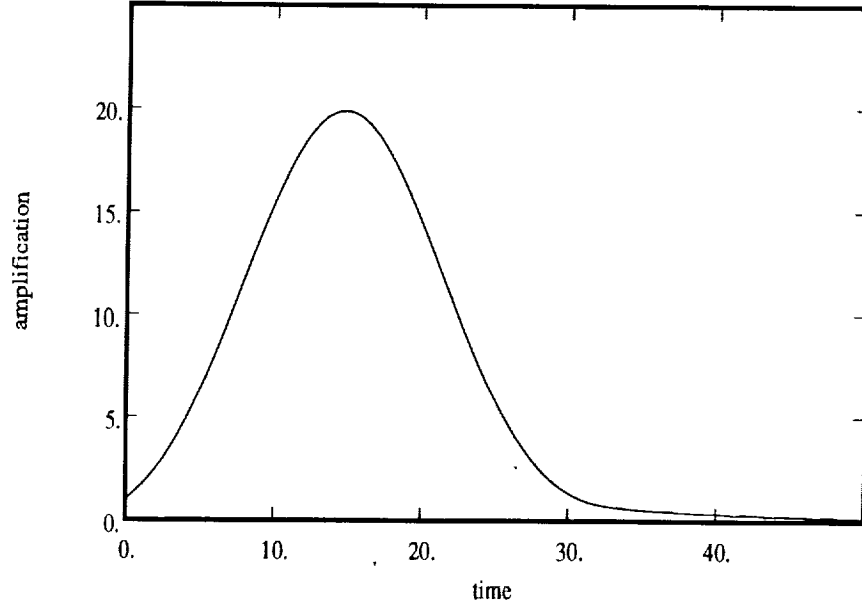


Fig.2. Plot of $\|\exp(-itL)\|$ versus time for plane Couette flow at $\alpha = 1$ and $R = 1500$.

The dependence of the sensitivity on the Reynolds number is shown in Table 2. For this case, the sensitivity has been defined as $-\log_{10}(\|(z_j I - L)^{-1}\|)$, i.e. the negative exponent of the resolvent norm evaluated at selected locations z_j in the complex plane. Table 2 shows a monotonic increase of the sensitivity with increasing Reynolds number which is more pronounced for locations near the intersection point of the three eigenvalue branches than near the least stable modes.

Reynolds number	sensitivity		
	$z = 0.7 - i 0.2$	$z = 0.4 - i 0.4$	$z = -i 0.4$
500	1.772	3.092	4.544
1000	2.095	4.343	5.277
1500	2.752	4.975	6.770
2000	2.850	5.790	7.232
2500	3.020	6.434	8.446

Table.2. Dependence of the sensitivity on the Reynolds number at selected locations in the complex plane.

3.2. Trailing Line Vortex Flow

Stability analysis of swirling flows is an active area of research. One particular mean flow which has received much attention in recent years is the trailing line (Batchelor) vortex model([9]-[12],[16]). In cylindrical coordinates (r, θ, z) the base flow for the columnar vortex is given by

$$U = 0, \quad V = \frac{q}{r}(1 - e^{-r^2}), \quad W = W_0 + e^{-r^2}, \quad (13)$$

where U , V , and W are the radial, tangential, and axial velocities, respectively. The swirl parameter q is related to the ratio of maximum azimuthal velocity to maximum axial velocity and W_0 represents the undisturbed outer axial flow. Without loss of generality, throughout this section we assume $W_0 = 1$.

The above vortical mean flow is perturbed by three-dimensional disturbances of the type

$$[u, v, w, p] = [\hat{u}(r, t), \hat{v}(r, t), \hat{w}(r, t), \hat{p}(r, t)] e^{i(\alpha z + n\theta)}, \quad (14)$$

where α is the axial wavenumber and n is the azimuthal wavenumber which takes on integer values only. As in the previous section, the linearized governing equations (see [9],[10]) are arranged in the form $\partial\phi/\partial t + i\mathcal{L}\phi = 0$, where ϕ is the state vector

$$\phi = [\hat{u}(r, t), \hat{v}(r, t), \hat{w}(r, t), \hat{p}(r, t)]^T. \quad (15)$$

The operator \mathcal{L} is given by

$$\mathcal{L} = \begin{pmatrix} l_{11} & (\frac{2V}{r} - \frac{2in}{Rr^2}) & 0 & -D \\ (DV - \frac{2in}{Rr^2} + \frac{V}{r}) & l_{22} & 0 & \frac{n}{r} \\ DW & 0 & l_{33} & \alpha \\ -\frac{i}{\gamma}(D + \frac{1}{r}) & -\frac{in}{\gamma r} & \frac{i\alpha}{\gamma} & 0 \end{pmatrix} \quad (16a)$$

with

$$l_{11} = l_{22} = \frac{iD}{R}(D + 1) + \frac{nV}{r} + \alpha W - \frac{i}{R}(\frac{n^2 + 1}{r^2} + \alpha^2)$$

and

$$l_{33} = l_{11} + \frac{i}{Rr^2}.$$

In equation (16a), D represents the radial derivative and R is the Reynolds number based on the viscous core radius and the axial velocity excess (or deficit).

The boundary conditions far from the centerline of the vortex are

$$\hat{u}(\infty) = \hat{v}(\infty) = \hat{w}(\infty) = 0. \quad (16b)$$

For single-valued and smooth solutions, the boundary conditions on the centerline of the vortex are

$$\begin{aligned} \text{if } n = 0, \quad & \hat{u}(0) = \hat{v}(0) = 0, \quad \hat{w}(0) \text{ and } \hat{p}(0) \text{ must be finite,} \\ \text{if } n = \pm 1, \quad & \hat{u}(0) \pm \hat{v}(0) = 0, \quad D\hat{u}(0) = 0, \quad \hat{w}(0) = \hat{p}(0) = 0 \\ \text{if } |n| > 1, \quad & \hat{u}(0) = \hat{v}(0) = \hat{w}(0) = \hat{p}(0) = 0. \end{aligned} \quad (16c)$$

The form of the continuity equation poses a problem as it does not contain a time dependent term and therefore cannot be cast into an eigenvalue problem. Although row and column operations can be used to solve this issue, a more efficient method, used in equation (16), is to artificially introduce the term $\gamma \partial p / \partial t$ to the continuity equation where the constant γ can be thought of as a weight parameter. This technique is known as the artificial compressibility method. For swirling flows, Khorrami, Malik & Ash[9] showed that for very small values of the constant γ , the physical eigenvalues remain unaffected and the spectrum is computed efficiently.

The operator \mathcal{L} is spatially discretized using a Chebyshev spectral collocation technique. The details of the discretization are given in [9] and are omitted here.

Two distinct stability characteristics have resulted in a classification of swirling stability modes into inviscid and viscous perturbations. Modes with negative azimuthal wavenumber are referred to as inviscid modes because their stability characteristics are determined by inviscid phenomena. The so-called unstable viscous modes are dominated by diffusive effects and include eigensolutions for azimuthal wavenumbers of $n = 0, 1$. For large Reynolds numbers $R \gg 1$, the growth rate of the viscous disturbances is inversely proportional to the Reynolds number, i.e. in the limit of infinite Reynolds number the viscous modes become neutrally stable. Based on previous investigations [10][11], the trailing line vortex represented by equation (13) is unstable with respect to both inviscid and viscous disturbances.

Due to space limitations, in this study we restrict our attention to a single azimuthal Fourier mode taken from the inviscid and viscous family. We chose parameter values that have been found to be the most interesting/challenging from a computational point of view.

Figure 3 displays the pseudospectrum for the inviscid case of trailing line vortex flow. The parameters are chosen as follows: $\alpha = 0.4, n = -1, R = 10^4, q = 0.4$ and $\gamma = 10^{-8}$. Figure 3a shows the ε -pseudospectrum for $\varepsilon = 10^{-5}$. Sensitive eigenvalues are observed close to the neutral line whereas the least stable modes show little effect to the perturbations imposed. Figure 3b displays the resolvent contours for the same case where the contour levels range from 10^{-6} to 10^{-4} .

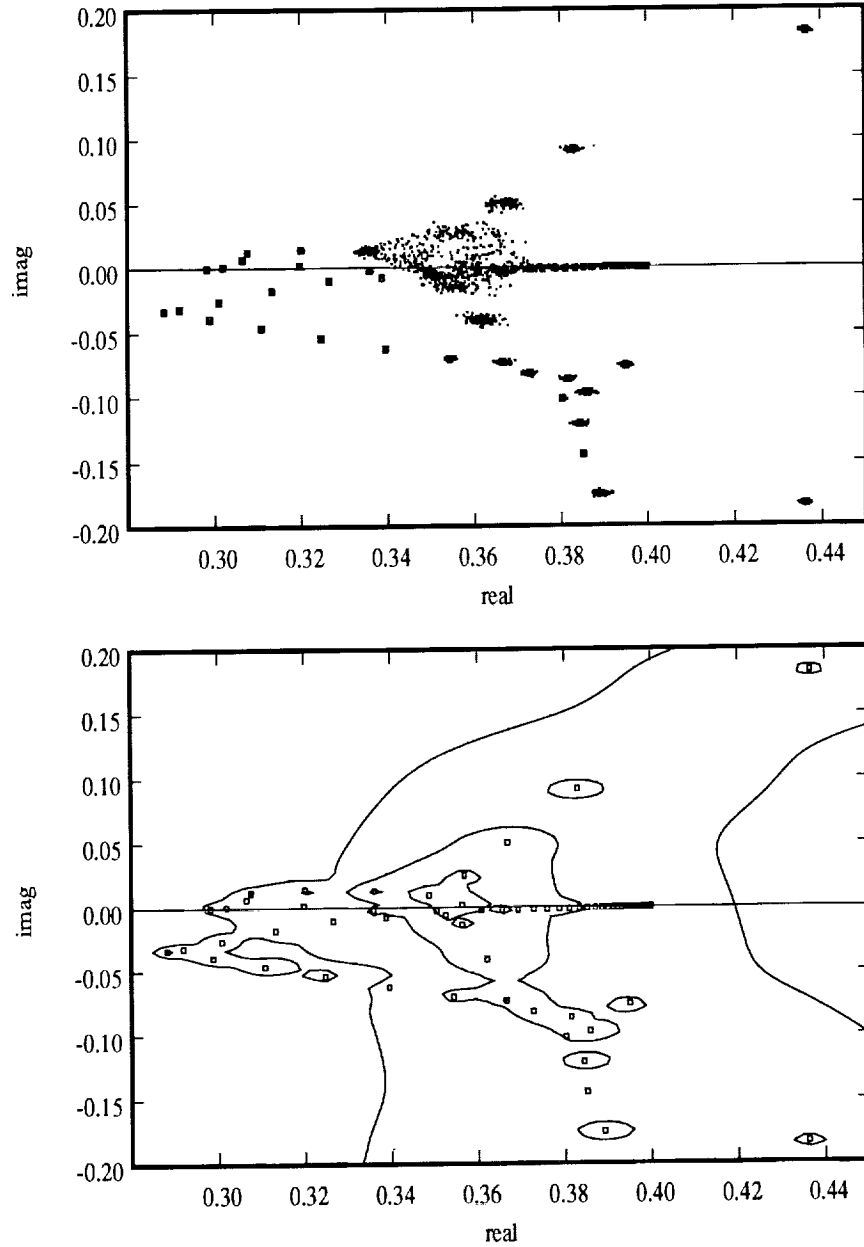


Fig.3. Pseudospectra for trailing line vortex flow (inviscid case) for $\alpha = 0.4, n = -1, R = 10^4, q = 0.4$ and $\gamma = 10^{-8}$. (a) Superposition of 100 spectra perturbed by random matrices of norm 10^{-5} . The unperturbed spectrum is displayed by the larger square symbols. (b) Contour plot of constant resolvent norm. The contours represent levels from 10^{-6} (innermost contour) to 10^{-4} (outer contour). Note the different scaling for the real and imaginary axis.

Table 3 lists the condition of selected eigenvalues and confirms the relative sensitivity shown in Figure 3.

As was the case for Couette flow, the sensitivity of the eigenvalues increases monotonically. Table 4 presents the sensitivity evaluated at three different locations in the complex plane as a function of Reynolds number. In all three cases we experience an increase in sensitivity with increasing Reynolds number.

λ_r	eigenvalue	condition
	λ_i	
0.436626	0.182285	$0.367540 \cdot 10^3$
0.383041	$0.914185 \cdot 10^{-1}$	$0.669828 \cdot 10^3$
0.366548	$-0.735235 \cdot 10^{-1}$	$0.602626 \cdot 10^3$
0.307724	$0.120135 \cdot 10^{-1}$	$0.830603 \cdot 10^2$
0.356920	$0.248804 \cdot 10^{-1}$	$0.400462 \cdot 10^4$

Table.3. Selected eigenvalues and conditions for swirling flow (inviscid case). Parameters as in caption of Figure 3.

Reynolds number	sensitivity		
	$z = 0.42 + i \cdot 0.12$	$z = 0.375 - i \cdot 0.03$	$z = 0.365 + i \cdot 0.02$
1000	4.076	4.176	5.236
2000	4.092	4.319	5.279
5000	4.111	4.683	5.322
10000	4.121	5.029	5.635
20000	4.128	5.228	5.968

Table.4. Dependence of the sensitivity on the Reynolds number (inviscid case).

The pseudospectrum for the viscous case of the trailing line vortex flow is shown in Figure 4 for $\alpha = 0.3, n = 1, R = 10^4, q = 0.4$ and $\gamma = 10^{-9}$. Figure 4a reveals that a perturbation of norm 10^{-6} results in a deviation of the eigenvalues orders of magnitude larger than the perturbation imposed. Figure 4b displays the contours of constant resolvent norm with levels ranging from 10^{-8} to 10^{-5} . It is interesting to note that the unstable eigenvalue lies within the 10^{-8} -contour.

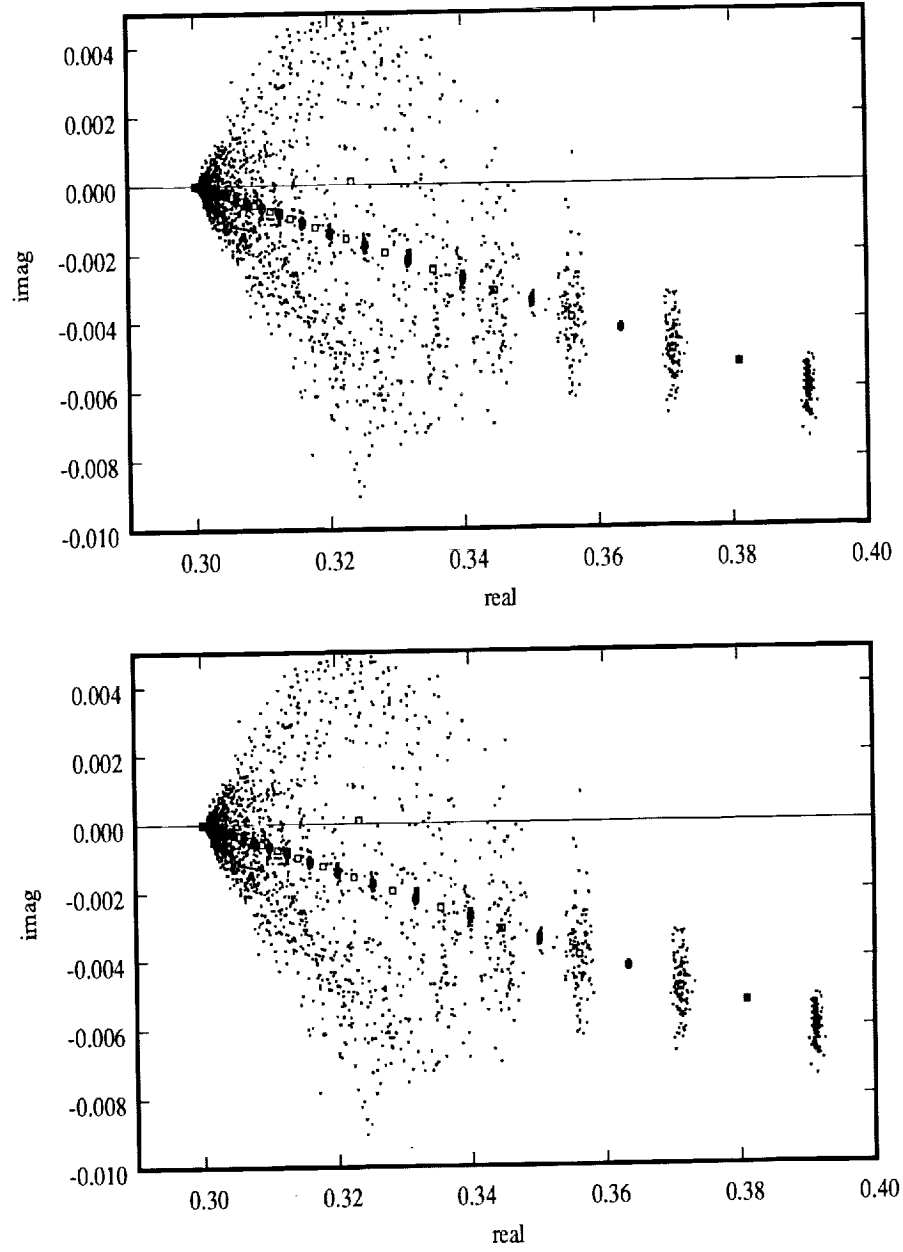


Fig.4. Pseudospectra for trailing line vortex flow (viscous case) for $\alpha = 0.3, n = 1, R = 10^4, q = 0.4$ and $\gamma = 10^{-9}$. (a) Superposition of 100 spectra perturbed by random matrices of norm 10^{-6} . The unperturbed spectrum is displayed by the larger square symbols. (b) Contour plot of constant resolvent norm. The contours represent levels from 10^{-8} (innermost contour) to 10^{-5} (outer contour). Note the different scaling for the real and imaginary axis.

Table 5, listing the condition for assorted eigenvalues, emphasizes this fact as the unstable mode shows the largest eigenvalue condition. A parameter study, summarized in Table 6, shows that the sensitivity of the spectrum, especially near the unstable mode, grows quite appreciably.

eigenvalue		condition
λ_r	λ_i	
0.380991	$-0.527545 \cdot 10^{-2}$	$0.231711 \cdot 10^3$
0.350173	$-0.339581 \cdot 10^{-2}$	$0.522167 \cdot 10^3$
0.323453	$0.918111 \cdot 10^{-4}$	$0.159909 \cdot 10^6$
0.322607	$-0.155546 \cdot 10^{-2}$	$0.150151 \cdot 10^6$
0.310078	$-0.655544 \cdot 10^{-3}$	$0.213876 \cdot 10^4$

Table.5. Selected eigenvalues and conditions for swirling flow (viscous case). Parameters as in caption of Figure 4.

Reynolds number	sensitivity		
	$z = 0.38 - i \cdot 0.004$	$z = 0.35 - i \cdot 0.002$	$z = 0.322 - i \cdot 0.001$
30	4.089	4.391	4.970
60	4.381	4.755	5.490
200	4.864	5.367	6.469
500	5.163	5.728	7.087
1000	5.362	5.946	7.408
3000	5.657	6.251	7.777
10000	5.772	6.407	8.220

Table.6. Dependence of the sensitivity on the Reynolds number

Sensitive eigenvalues close to the neutral line can result in large transient growth of the perturbation amplitude. In order to show that, we calculated the norm of the matrix exponential as described in section 2. Figure 7 shows the amplification factor $\|\phi\|/\|\phi_0\|$ versus time. A large transient growth is observed which only for large times is governed by the unstable eigenvalue.

The maximum imaginary part of the field of values has been determined to be 24.78 which confirms the large growth of the state vector norm as plotted in Fig.5.

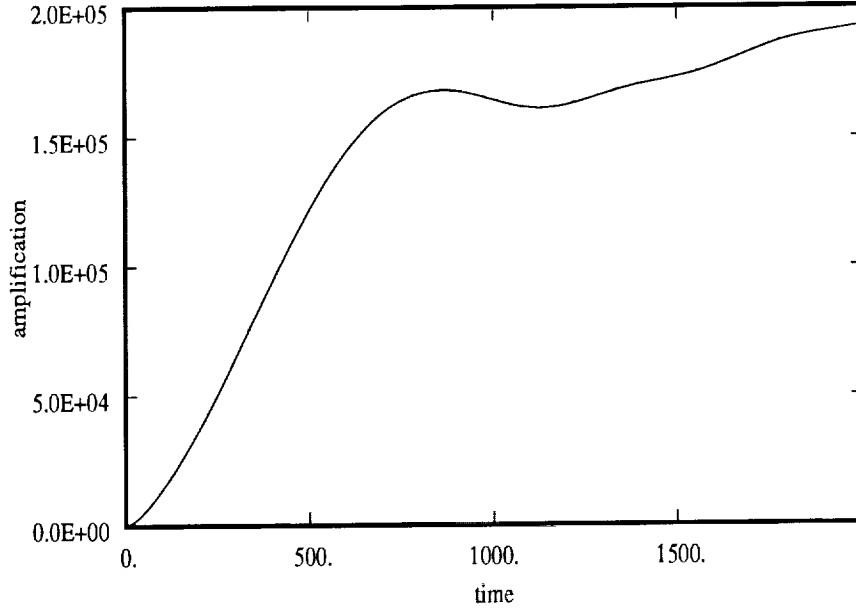


Fig.5. Plot of $\|\exp(-itL)\|$ versus time for trailing line vortex flow (viscous case) for parameters given in caption of Figure 4.

3.3. Compressible Blasius Boundary Layer Flow

In this section we investigate the *spatial* evolution of infinitesimal disturbances as opposed to the temporal evolution in the previous examples.

Consider compressible boundary-layer flow past a flat-plate. It is convenient to write the basic state using the similarity transformation

$$d\eta = \sqrt{\frac{\rho_e u_e}{2\mu_e x}} (\rho/\rho_e) dy \quad (17)$$

where x, y are the coordinates parallel and normal to the plate, ρ_e is the boundary-layer edge density, u_e the streamwise velocity and μ_e the dynamic viscosity. The governing equations for the mean flow can be written in a similarity form as

$$(cf'')' + ff'' = 0 \quad (18)$$

$$(a_1 g' + a_2 f' f'')' + fg' = 0 \quad (19)$$

where

$$f' = u/u_e, \quad c = \rho\mu/\rho_e\mu_e$$

$$g = H/H_e, \quad a_1 = c/\sigma, \quad a_2 = \frac{(\gamma - 1)M^2}{1 + (\frac{\gamma - 1}{2})M^2} \left(1 - \frac{1}{\sigma}\right)c$$

and M is the Mach number, γ the ratio of specific heats, σ the Prandtl number, and H the total enthalpy.

The compressible boundary-layer flow is assumed to be locally parallel and is perturbed by a harmonic disturbance of the form

$$\Phi = \hat{\Phi}(x, y)e^{i(\beta z - \omega t)} \quad (20)$$

where β is the spanwise wavenumber and ω is the disturbance frequency. We consider the spatial stability in which disturbances grow in the streamwise direction x . The spatial eigenvalue problem is nonlinear due to the streamwise viscous diffusion ($\hat{\Phi}_{xx}$) term. It was argued in [15] that the eigenvalue problem can be linearized by dropping the $\hat{\Phi}_{xx}$ term since the instability in supersonic flows is predominantly inviscid. Later work suggests that this approximation is valid even at lower Mach numbers and is also consistent with the triple-deck description of the Tollmien-Schlichting wave structure. Hence, the governing compressible linear stability equations can be written as

$$\frac{\partial \hat{\Phi}}{\partial x} + i\mathcal{L}\hat{\Phi} = 0 \quad (21)$$

where $\hat{\Phi}$ is a five-element vector defined by $\{\hat{\Phi}_1, \hat{\Phi}_2, \hat{\Phi}_3, \hat{\Phi}_4, \hat{\Phi}_5\}^T = \{\hat{u}, \hat{v}, \hat{p}, \hat{T}, \hat{w}\}^T$, and the linear operator \mathcal{L} is of the form

$$\mathcal{L} = A \frac{d^2}{dy^2} + B \frac{d}{dy} + C \quad (22)$$

where A, B and C are 5×5 matrices whose elements are related to the matrices given in [15].

Equation (21) is solved subject to the boundary conditions

$$\hat{\Phi}_1 = \hat{\Phi}_2 = \hat{\Phi}_4 = \hat{\Phi}_5 = 0 \quad \text{at } y = 0, \quad (23a)$$

$$\hat{\Phi}_1, \hat{\Phi}_2, \hat{\Phi}_4, \hat{\Phi}_5 \rightarrow 0 \quad \text{as } y \rightarrow \infty. \quad (23b)$$

In the spatial analysis, the x -dependence of the disturbance is expressed in the form $\exp(i\lambda x)$ which results in the eigenvalue problem $-\lambda\hat{\Phi} = \mathcal{L}\hat{\Phi}$. Disturbances are expected to grow in x if $\text{Im}(\lambda) < 0$.

The operator \mathcal{L} is discretized by a Chebyshev spectral collocation technique described in [15].

The pseudospectrum for compressible boundary layer flow is computed for a Reynolds number $R = 1000$, where $R = \sqrt{\rho_e u_e x / \mu_e}$. In Fig.6, it is displayed for the parameter

combination $M = 2, \beta = 0.08$ and $\omega = 0.02$. Figure 6a shows a superposition of 100 spectra perturbed by random matrices of norm 10^{-3} . Although the sensitivity of the eigenvalues is less pronounced than in the previous cases, parts of the spectrum do show eigenvalues that deviate orders of magnitude more than the perturbation that caused the deviation. Especially the least stable mode and the continuous entropy branch exhibit a marked response to the perturbations as displayed in Figure 6b. This observation is again confirmed by calculating the condition of the respective eigenvalues (Table 7).

eigenvalue		condition
λ_r	λ_i	
$0.407748 \cdot 10^{-1}$	$-0.239187 \cdot 10^{-2}$	$0.210794 \cdot 10^2$
$0.200015 \cdot 10^{-1}$	$0.115666 \cdot 10^{-1}$	$0.204110 \cdot 10^1$
$0.200007 \cdot 10^{-1}$	$0.460632 \cdot 10^{-1}$	$0.190301 \cdot 10^1$
$0.529917 \cdot 10^{-1}$	$0.338088 \cdot 10^{-1}$	$0.425356 \cdot 10^2$
0.108886	$0.178297 \cdot 10^{-3}$	$0.228352 \cdot 10^1$

Table.7. Selected eigenvalues and conditions for compressible boundary layer flow. Parameters as in caption of Figure 6.

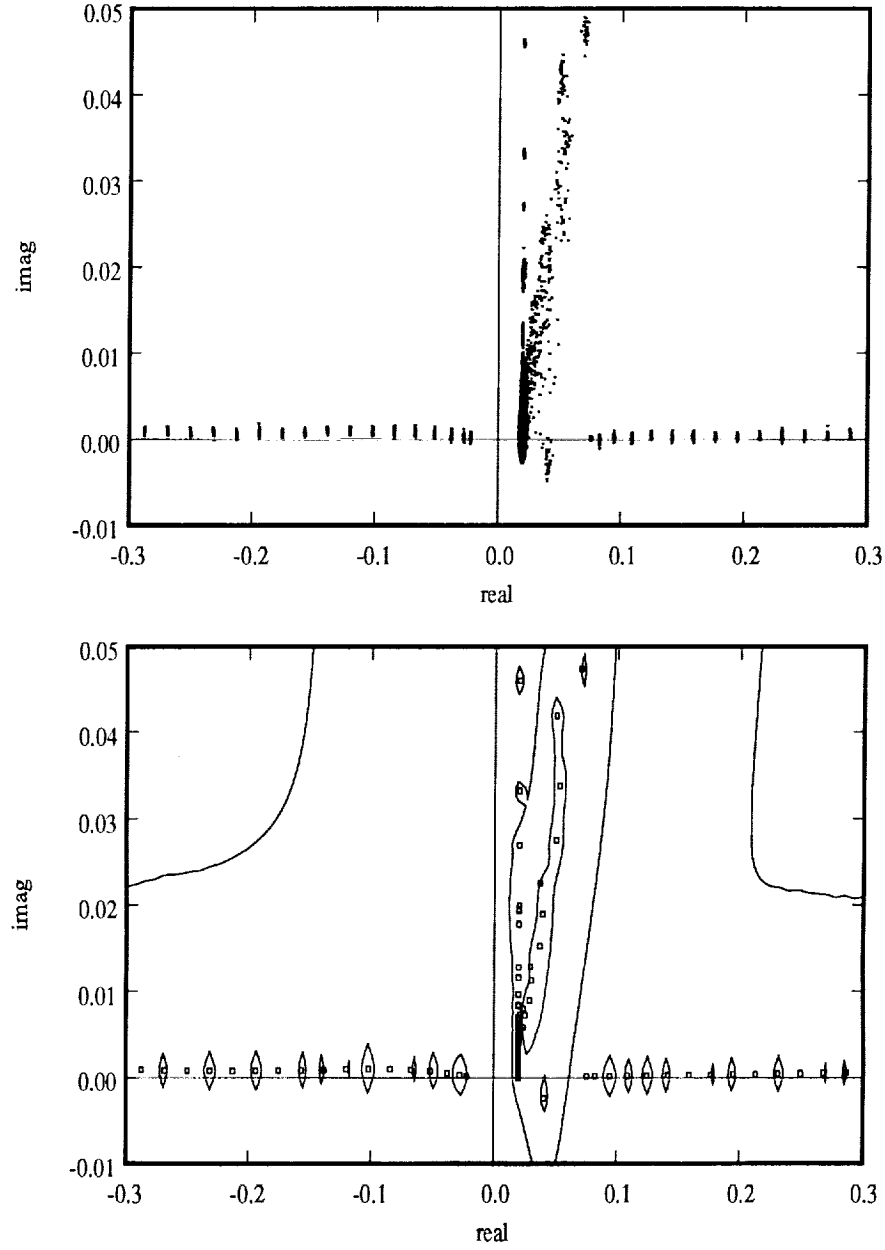


Fig.6. Pseudospectra for compressible boundary layer flow for $M = 2$, $\beta = 0.08$, $R = 1000$ and $\omega = 0.02$. (a) Superposition of 100 spectra perturbed by random matrices of norm 10^{-3} . (b) Contour plot of constant resolvent norm. The contours represent levels from 10^{-4} (innermost contour) to 10^{-2} (outer contour). The spectrum is displayed by the square symbols. Note the different scaling for the real and imaginary axis.

4. Summary and Conclusions

A sensitivity study of hydrodynamic stability operators has been conducted to investigate the response of eigenvalues to small perturbations. Classical techniques from matrix perturbation theory as well as the concept of ε -pseudoeigenvalues has been applied to obtain qualitative and quantitative results. Applications involved incompressible plane Couette flow, incompressible trailing line vortex flow and compressible Blasius boundary layer flow. All examples showed sensitive eigenvalues that deviated from their unperturbed location by orders of magnitude more than the norm of the perturbations imposed on the matrix entries.

In the case of Couette flow, the most sensitive eigenvalues have been found in the junction of the three eigenvalue branches whereas the least stable eigenvalues showed a rather small response to small random perturbations. The sensitivity of the spectrum measured by the norm of the resolvent evaluated at specific points in the complex plane increased monotonically with Reynolds number.

For the case of trailing line vortex flow, viscous and inviscid perturbations have to be distinguished. For inviscid perturbations, again the unstable mode seems to be rather insensitive and the regions of high sensitivity are located close to the real axis. The case of viscous disturbances imposed on the vortex flow, however, showed a dramatic effect of small perturbations on the location of the eigenvalues. The only unstable mode experienced the highest sensitivity implying the loss of its predictive character because of the dependence on very small perturbations. As in Couette flow, in both cases of trailing line vortex flow the sensitivity increased monotonically with Reynolds number. Investigations of the norm of the matrix exponential revealed large transient growth which resulted in a growth rate superior to the one predicted by the unstable eigenvalue. The field of values of the linear operator gave a necessary and sufficient condition for transient growth as well as a measure for the maximum possible growth rate.

Although the compressible Blasius boundary layer spectrum appeared to be less sensitive to small perturbations than the previous cases, only a small perturbation is needed to find eigenvalues stemming from the continuous entropy branch in the unstable half-plane. It is conjectured that the sensitivity will increase for higher Reynolds and/or Mach numbers.

One of the main interest of hydrodynamic stability theory is the understanding of the temporal behavior of small perturbations superimposed on a steady flow field. This behavior is governed by the initial-boundary-value problem stated in section 1 of this presentation. The analysis can, therefore, be restricted to understanding the behavior of the operator \mathcal{L} .

There are two distinct regimes to be considered: the asymptotic regime of large times ($t \rightarrow \infty$) and the transient regime. In the first regime, it has been shown that the eigenvalues of \mathcal{L} are sufficient to completely describe the behavior of \mathcal{L} . In the second regime, however, eigenvalues do not suffice to describe the temporal behavior since it may not be mode-like. Hence, there is a need to replace eigenvalue analysis by a more

effective tool to analyze operators which can experience large growth due to their non-normality during this transient period. ε -pseudoeigenvalues, i.e. eigenvalues of the slightly perturbed operator, can be used to extract informations about the operator in its transient phase and to “visualize its behavior”.

As the transient growth can be quite substantial, the nonlinear regime may be reached during the transient phase, rather than by a mechanism based on exponential growth. As has been shown [20], the growth mechanism stemming from the non-normality of the linear operator plays a dominant role during the early stages of transition to turbulence in parallel shear flow. For incompressible flows it is only this linear mechanism that can provide an increase in total perturbation energy. The implication of growth due to non-normality is therefore of primary importance in hydrodynamic stability theory. A necessary condition for growth is the non-normality of the stability operator and sufficient conditions can be derived (together with bounds on the maximum growth possible) from the ε -pseudospectrum [19].

For these reasons, the study of the ε -pseudospectrum is essential when encountering highly non-normal operators or matrices.

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